Polynomial τ-functions of the NLS-Toda hierarchy and the Virasoro singular vectors

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To Minoru Wakimoto on his sixtieth birthday

Abstract

A family of polynomial τ-functions for the NLS-Toda hierarchy is constructed. The hierarchy is associated with the homogeneous vertex operator representation of the affine algebra $\mathfrak{g}$ of type $A_1^{(1)}$. These τ-functions are given explicitly in terms of Schur functions that correspond to rectangular Young diagrams. It is shown that an arbitrary polynomial τ-function which is an eigenvector of d, the degree operator of $\mathfrak{g}$, is contained in the family. By the construction, any τ-function in the family becomes a Virasoro singular vector. This consideration gives rise to a simple proof of known results on the Fock representation of the Virasoro algebra with $c = 1$.

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1 Introduction

The main object of this paper is the family of the Schur functions associated with the rectangular Young diagrams. We have encountered these *rectangular Schur functions* in the studies of integrable systems and related representation theories:

- The Virasoro singular vectors in the Fock space of central charge $c = 1$ [1],[2],
- Polynomial τ-functions of the NLS-Toda hierarchy [3],[4].

One of the aim of this paper is to understand these two results in a unified manner. Another aim is to parametrize all the τ-functions of the NLS-Toda hierarchy which are homogeneous polynomials with respect to the degree specified below. As the common background, we utilize the homogeneous vertex operator construction of the basic representation $L(\Lambda_0)$ of the affine Lie algebra $\mathfrak{g}(A_1^{(1)})$ ([5]).

Among many ways of introducing the NLS-Toda hierarchy, our main viewpoint is group-theoretic ([6],[8],[7]). One knows that the basic representation $L(\Lambda_0)$ is realized on the space $V \stackrel{\text{def}}{=} \mathbb{C}[\boldsymbol{t}\,] \otimes \mathbb{C}[e^{\alpha}, e^{-\alpha}]$, where $\boldsymbol{t} = (t_1, t_2, \dots)$. The NLS-Toda hierarchy characterizes the G-orbit of the highest weight vector in V, where we denote by G the group associated with $\mathfrak{g}(A_1^{(1)})$ introduced by Peterson and Kac [9]. The hierarchy is also described in terms of Hirota's bilinear forms. It is given by infinite number of bilinear equations including the following typical equations:

$$(D_{t_1}^2 + D_{t_2})\tau_n \cdot \tau_{n+1} = 0, \quad D_{t_1}^2 \tau_n \cdot \tau_n + 2\tau_{n-1}\tau_{n+1} = 0 \quad (n \in \mathbb{Z}), \tag{1.1}$$

where we write a typical element τ of V as $\tau = \sum_{n\in\mathbb{Z}} \tau_n e^{n\alpha}$ ($\tau_n \in \mathbb{C}[\boldsymbol{t}\,]$). The second equation of (1.1) is known as the bilinear form the Toda lattice equation. If we set $\Phi \stackrel{\text{def}}{=} \tau_{n+1}/\tau_n, \overline{\Phi} \stackrel{\text{def}}{=} \tau_{n-1}/\tau_n$ for any fixed n and pose the assumption that $\overline{\Phi}$ is the complex conjugate of $\pm\Phi$, then, from equations (1.1), we have

$$-\, i\partial_t \Phi = \partial_x^2 \Phi \mp 2|\Phi|^2 \Phi, \tag{1.2}$$

where $t \stackrel{\text{def}}{=} it_2$ and $x \stackrel{\text{def}}{=} t_1$. Equation (1.2) is known as the nonlinear Schrödinger (NLS) equation. So we call this hierarchy as the NLS-Toda hierarchy. Let us define the degree in the space V by

$$\deg(t_j) = j (j \geq 1), \quad \deg(e^{n\alpha}) = n^2 (n \in \mathbb{Z}). \tag{1.3}$$

A direct computation of the $\mathrm{SL}_2(\mathbb{C})$-orbit of $e^{n\alpha}$ leads to a formula (4.26) of τ-functions of the hierarchy. The τ-functions are written in terms of rectangular Schur functions. We can also prove that the family given by (4.26) exhaust all the homogeneous τ-functions of the hierarchy.

One knows that the Virasoro algebra acts on the space V, where the each sector $\mathbb{C}[\boldsymbol{t}\,]e^{n\alpha} (n \in \mathbb{Z})$ is preserved. It is remarkable that the Virasoro algebra commutes with the Lie subalgebra $\mathfrak{sl}_2(\mathbb{C})$ of $\mathfrak{g}(A_1^{(1)})$. Since our τ-functions are given as $\mathrm{SL}_2(\mathbb{C})$-orbit through the vector $e^{n\alpha}$, it is immediate to see that the τ-function is a singular vector,

that is we have $L_k\tau = 0$ for $k > 0$. Hence we have a simple explanation of the result of Segal [1] and Wakimoto-Yamada [2].

We here add some historical remarks. It should be firstly mentioned that Sachs [3] obtained the formula for τ-functions in terms of rectangular Schur functions by using the Jacobi identity. Gilson et al. [4] also derived the formula from the double Wronskian type solution to the 2-component KP hierarchy. In addition, M. Sato made the following inspiring comment, unfortunately without any proof, in his series of lectures delivered at Kyoto University, : "The whole family of the rectangular Schur functions characterizes the NLS hierarchy".

2 Homogeneous construction of the basic representation

First we recall some facts on the basic representation $L(\Lambda_0)$ of the affine Lie algebra $\mathfrak{g}$ of type $A_1^{(1)}$ due to Frenkel and Kac (cf. [5], see also [10]).

Let $\overset{\circ}{\mathfrak{g}} = \mathfrak{sl}(2,\mathbb{C})$ with the standard basis

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

The affine Lie algebra $\mathfrak{g}$ of type $A_1^{(1)}$ can be realized as the vector space

$$\mathfrak{g} = \overset{\circ}{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d \tag{2.5}$$

with the bracket $[K, \mathfrak{g}] = 0,\ [d, X \otimes t^n] = nX \otimes t^n$, and

$$[X \otimes t^m, Y \otimes t^n] = [X, Y] \otimes t^{m+n} + m\delta_{m+n,0}\mathrm{tr}(XY)K \quad (X, Y \in \overset{\circ}{\mathfrak{g}}, m, n \in \mathbb{Z}) \tag{2.6}$$

where $[X, Y]$ is the bracket in $\overset{\circ}{\mathfrak{g}}$. For $X \in \overset{\circ}{\mathfrak{g}}, n \in \mathbb{Z}$ put $X_n \overset{\mathrm{def}}{=} X \otimes t^n$. We identify $\overset{\circ}{\mathfrak{g}}$ with the subalgebra of $\mathfrak{g}$ consisting of the elements $X_0 (X \in \overset{\circ}{\mathfrak{g}})$.

Consider the subalgebra $\mathfrak{h} = \mathbb{C}H \oplus \mathbb{C}K \oplus \mathbb{C}d$. It is called a *Cartan subalgebra* of $\mathfrak{g}$. Let Λ_0, δ be the linear functions on $\mathfrak{h}$ defined by

$$\Lambda_0(H) = 0,\ \Lambda_0(K) = 1,\ \Lambda_0(d) = 0,\ \delta(H) = \delta(K) = 0,\ \delta(d) = 1. \tag{2.7}$$

The *basic representation* $L(\Lambda_0)$ of the Lie algebra $\mathfrak{g}$ is the irreducible highest weight representation of the highest weight Λ_0. Namely it is an irreducible $\mathfrak{g}$-module in which there exists a non-zero vector v_{Λ_0} such that

$$Kv_{\Lambda_0} = v_{\Lambda_0}, \quad dv_{\Lambda_0} = 0, \quad X_n v_{\Lambda_0} = 0\ (n \geq 0). \tag{2.8}$$

We have

$$[H_m, H_n] = 2m\delta_{m+n,0}K. \tag{2.9}$$

The elements $H_n (n \neq 0)$ generate a subalgebra $\mathcal{H}$, which is called the *homogeneous* Heisenberg subalgebra of $\mathfrak{g}$. Let $\boldsymbol{t}$ denote the sequence of infinitely many independent variables $t_1, t_2, \dots$. We have a natural representation of $\mathcal{H}$ on the space of polynomials $\mathbb{C}[\boldsymbol{t}\,]$ given by

$$H_n \mapsto 2\frac{\partial}{\partial t_n}, \quad H_{-n} \mapsto n t_n \ (n > 0), \quad K \mapsto \mathrm{id}. \tag{2.10}$$

We construct a representation of $\mathfrak{g}$ on the space $V \stackrel{\text{def}}{=} \mathbb{C}[\boldsymbol{t}\,] \otimes \mathbb{C}[Q]$, where $\mathbb{C}[Q]$ is the group algebra of the root lattice $Q \stackrel{\text{def}}{=} \mathbb{Z}\alpha$ of $\mathring{\mathfrak{g}}$. Let the elements of $\mathcal{H}$ act on the first factor of $\mathbb{C}[\boldsymbol{t}\,] \otimes \mathbb{C}[Q]$ by (2.10). Define also an action of $H = H_0$ on V by

$$H(P \otimes e^{n\alpha}) = 2nP \otimes e^{n\alpha} \ (P \in \mathbb{C}[\boldsymbol{t}\,], n \in \mathbb{Z}). \tag{2.11}$$

To describe the actions of E_n and F_n, it is convenient to consider the generating function

$$X(z) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} X_n z^{-n-1} \quad (X \in \mathring{\mathfrak{g}}). \tag{2.12}$$

Let us introduce the following operators on V:

$$z^{\pm H}(P \otimes e^{n\alpha}) = z^{\pm 2n} P \otimes e^{n\alpha}, \quad e^{\pm\alpha}(P \otimes e^{n\alpha}) = P \otimes e^{(n\pm 1)\alpha} \quad (P \in \mathbb{C}[\boldsymbol{t}\,],\ n \in \mathbb{Z}). \tag{2.13}$$

Then the action of $E(z)$ and $F(z)$ are given by the following *vertex operators*:

$$E(z) \mapsto e^{\eta(\boldsymbol{t},z)} e^{-2\eta(\tilde{\partial}_{\boldsymbol{t}},z^{-1})} e^{\alpha} z^{H}, \quad F(z) \mapsto e^{-\eta(\boldsymbol{t},z)} e^{2\eta(\tilde{\partial}_{\boldsymbol{t}},z^{-1})} e^{-\alpha} z^{-H} \tag{2.14}$$

where

$$\eta(\boldsymbol{t}, z) = \sum_{j=1}^{\infty} t_j z^j, \quad \eta(\tilde{\partial}_{\boldsymbol{t}}, z^{-1}) = \sum_{j=1}^{\infty} \frac{1}{j}\frac{\partial}{\partial t_j} z^{-j}. \tag{2.15}$$

If we set $v_{\Lambda_0} \stackrel{\text{def}}{=} 1 \otimes e^0 \in V$ then we have (2.8). It is shown that the set of formulas (2.10),(2.11), (2.13), and (2.14) gives an irreducible representation of $\mathfrak{g}$ on V. In this way, we obtain a realization of $L(\Lambda_0)$ on the space V.

3 Definition of the NLS-Toda hierarchy

Since $L(\Lambda_0) = V$ is an *integrable* representation, we can define the actions of $\exp(E_n), \exp(F_n)$ $(n \in \mathbb{Z})$ on V. These operators generate a subgroup of $\mathrm{GL}(V)$ which we denote by G. Let $\mathcal{O} = G v_{\Lambda_0}$ be the G-orbit through the highest weight vector v_{Λ_0}.

Definition 1 *A non-zero vector $\tau \in V = \mathbb{C}[\boldsymbol{t}] \otimes \mathbb{C}[Q]$ is called a τ-function of the NLS-Toda hierarchy if and only if $\tau \in \mathcal{O}$.*

Let L_{high} be the highest component of $V \otimes V$, i.e., the $\mathfrak{g}$-submodule in $V \otimes V$ generated by $v_{\Lambda_0} \otimes v_{\Lambda_0}$. Then we have $L_{high} \cong L(2\Lambda_0)$ and the group orbit $\mathcal{O}$ can also be described as follows:

$$\tau \in \mathcal{O} \Longleftrightarrow \tau \otimes \tau \in L_{high}. \tag{3.16}$$

Further, the method of the *generalized Casimir operator* due to Kac and Wakimoto ([11]) enables us to write down the condition $\tau \otimes \tau \in L_{high}$ as a system of Hirota's bilinear differential equations. In particular, a τ-function of the NLS-Toda hierarchy satisfies (1.1). In this paper, however, we do not use the explicit forms of these equations. The bilinear equations are also derived from the free fermion operators and the boson-fermion correspondence (see [6],[8]).

4 Rectangular Schur functions

In this section we will show that the rectangular Schur functions appear naturally in the $\mathrm{SL}_2(\mathbb{C})$-orbit through the maximal weight vector of $L(\Lambda_0)$.

Let the polynomials $p_k(\boldsymbol{t})$ be defined by

$$e^{\eta(\boldsymbol{t},z)} = \sum_{k=0}^{\infty} p_k(\boldsymbol{t}) z^k, \tag{4.17}$$

where $\eta(\boldsymbol{t}, z) = \sum_{j=1}^{\infty} t_j z^j$. Then the Schur function S_λ indexed by the partition $\lambda = (\lambda_1, \dots, \lambda_n)$ can be expressed by the following determinant:

$$S_\lambda = S_\lambda(\boldsymbol{t}) = \det \left(p_{\lambda_i - i + j}(\boldsymbol{t})\right)_{1 \le i,j \le n}, \tag{4.18}$$

where we agree that $p_k(\boldsymbol{t}) = 0$ for $k < 0$. Let λ' be the conjugate of λ. Then we have

$$S_\lambda(-\boldsymbol{t}) = (-1)^{|\lambda|} S_{\lambda'}(\boldsymbol{t}), \quad |\lambda| = \sum_{j=1}^{n} \lambda_j. \tag{4.19}$$

If $\boldsymbol{t}$ is expressed by a sequence of finite number of variables $\boldsymbol{z} = (z_1, \dots, z_n)$ as

$$t_j = \frac{1}{j}\left(z_1^j + \cdots + z_n^j\right) \quad (j \ge 1) \tag{4.20}$$

then we have

$$S_\lambda = S_\lambda(\boldsymbol{z}) = \frac{\det(z_i^{\lambda_j + n - j})_{1 \le i,j \le n}}{\det(z_i^{n-j})_{1 \le i,j \le n}}. \tag{4.21}$$

The denominator is nothing but the Vandermonde determinant:

$$\Delta_n(\boldsymbol{z}) := \prod_{i<j} (z_i - z_j). \tag{4.22}$$

As for the rectangular Young diagram $(k, \ldots, k)$ (repeated n times) denoted by $\square(n,k)$, we see from (4.21) that $S_{\square(n,k)}(\boldsymbol{z})$ has a simple form

$$S_{\square(n,k)}(\boldsymbol{z}) = (z_1 \cdots z_n)^k. \tag{4.23}$$

Let $\boldsymbol{t} = (t_1, t_2, \ldots)$ and $\boldsymbol{z} = (z_1, \ldots, z_n)$ be two sequences of infinite and finite variables, respectively. The Cauchy formula (see [12]) can be written as

$$e^{\sum_{k=1}^n \eta(\boldsymbol{t}, z_k)} = \sum_\lambda S_\lambda(\boldsymbol{t}) S_\lambda(\boldsymbol{z}), \tag{4.24}$$

where summation runs over partitions λ with length at most n.

We also note orthogonality relations for the $S_\lambda(\boldsymbol{z})$. If $f = f(z_1, \ldots, z_n)$ is a Laurent polynomial, let $\overline{f} = f(z_1^{-1}, \ldots, z_n^{-1})$ and let $CT[f]$ denote the constant term in f. A scalar product is defined by $\langle f, g\rangle_n = \frac{1}{n!} CT[f\overline{g}\Delta_n\overline{\Delta_n}]$. Then we have

$$\langle S_\lambda, S_\mu\rangle_n = \delta_{\lambda\mu}. \tag{4.25}$$

Our main result in this section is the following.

Theorem 1 *Let k be a non-negative integer, and $\gamma \in \mathbb{C}$. We have the following expression of a τ-function:*

$$\exp(\gamma F)\left(e^{k\alpha}\right) = \sum_{n=0}^{2k} (-1)^{\frac{n(n+1)}{2}} \gamma^n S_{\square(2k-n,n)}(\boldsymbol{t}) e^{(k-n)\alpha}. \tag{4.26}$$

Proof. First we note that

$$F(z_1) \cdots F(z_n) e^{k\alpha} = e^{-\sum_{j=1}^n \eta(\boldsymbol{t}, z_j)} (z_1 \cdots z_n)^{-2k} \Delta_n^2 \, e^{(k-n)\alpha}. \tag{4.27}$$

It can be verified by using the relations

$$e^{-\eta(\boldsymbol{t},z)} e^{-2\eta(\tilde{\partial}_{\boldsymbol{t}}, w^{-1})} = \left(1 - \frac{w}{z}\right)^2 e^{-2\eta(\tilde{\partial}_{\boldsymbol{t}}, w^{-1})} e^{-\eta(\boldsymbol{t},z)}, \quad z^{-H} q^{-1} = z^2 q^{-1} z^{-H}. \tag{4.28}$$

Now we want to pick up the coefficient of $(z_1 \cdots z_n)^{-1}$ in (4.27), which is equal to $F^n e^{k\alpha}$. Notice that the coefficient can be written in terms of the scalar product

$$\begin{aligned} F^n e^{k\alpha} &= CT[z_1 \cdots z_n F(z_1) \cdots F(z_n) e^{k\alpha}] \\ &= \sum_\lambda (-1)^{\frac{n(n-1)}{2}} S_\lambda(-\boldsymbol{t}) n! \langle S_\lambda, S_{\square(n,2k-n)}\rangle_n e^{(k-n)\alpha}, \end{aligned} \tag{4.29}$$

where we expand $e^{-\sum_{j=1}^n \eta(\boldsymbol{t}, z_j)}$ by (4.24) and use $\overline{\Delta_n} = (-1)^{\frac{n(n-1)}{2}} (z_1 \cdots z_n)^{-n+1} \Delta_n$. Using the orthogonality (4.25) and (4.19) we have

$$F^n e^{k\alpha} = n! (-1)^{\frac{n(n-1)}{2}} S_{\square(n,2k-n)}(-\boldsymbol{t}) = n! (-1)^{\frac{n(n+1)}{2}} S_{\square(2k-n,n)}(\boldsymbol{t}). \tag{4.30}$$

Q.E.D.

5 Parametrization of the homogeneous τ-functions

Definition 2 *If $\tau \in \mathcal{O}$ is an eigenvector of d with eivenvalue N then we say that τ is homogeneous of degree N.*

We first state the following theorem that claims the family of τ-functions constructed in Section 4 exhausts all the homogeneous polynomial τ-functions.

Theorem 2 *Let τ be a homogeneous τ-function of degree N. Then there exists a non-negative integer m such that $N = m^2$. If, moreover, τ is not constant multiple of $e^{\pm m\alpha}$, then there exist constants $\gamma, c \neq 0$ such that $\tau = c\exp(\gamma F)\, e^{m\alpha}$.*

To prove Theorem 2, we will use a fundamental result by Peterson-Kac. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody Lie algebra, and $\mathfrak{h}$ be its Cartan subalgebra. Let $\Lambda \in \mathfrak{h}^*$ be a dominant integral weight and $L(\Lambda)$ be an irreducible $\mathfrak{g}$-module with highest weight Λ. We denote by v_Λ a highest weight vector in $L(\Lambda)$. For $v \in L(\Lambda)$ let $v = \sum_{\lambda \in \mathfrak{h}^*} v_\lambda$ be its weight decomposition. Put $\mathrm{supp}(v) \stackrel{\mathrm{def}}{=} \{\lambda \in P(\Lambda) \mid v_\lambda \neq 0\}$, and let $S(v)$ be the convex hull of $\mathrm{supp}(v)$ in $\mathfrak{h}^*$. By $G \subset \mathrm{GL}(L(\Lambda))$ we denote the associated group. And let W denote the Weyl group of $\mathfrak{g}$.

Lemma 1 *[9] Let $\tau \in L(\Lambda)$ be in the G-orbit through v_Λ. Then each vertex of $S(\tau)$ belongs to the W-orbit of the highest weight Λ.*

Proof of Theorem 2. For $\mathfrak{g} = \mathfrak{g}(A_1^{(1)})$, W-orbit of Λ_0 consists of the elements

$$\lambda_m \stackrel{\mathrm{def}}{=} \Lambda_0 + m\alpha - m^2\delta \ (m \in \mathbb{Z}). \tag{5.31}$$

Note that λ_m is the weight of $e^{m\alpha} \in V$. Let $\tau \in \mathcal{O}$ be homogeneous of degree N. By applying Lemma 1, we see that there exists a non-negative integer m such that $N = m^2$ and that $S(\tau)$ is the segment

$$[\lambda_m, \lambda_{-m}] \stackrel{\mathrm{def}}{=} \{s\lambda_m + (1-s)\lambda_{-m} \mid 0 \leq s \leq 1\} \tag{5.32}$$

unless $S(\tau) = \{\lambda_m\}$ or $\{\lambda_{-m}\}$. If $S(\tau) = \{\lambda_{\pm m}\}$ then τ is a constant multiple of $e^{\pm m\alpha}$ respectively. Therefore we assume $S(\tau) = [\lambda_m, \lambda_{-m}]$ and write $\tau = \sum_{n=0}^{2m} \tau_{m-n} e^{(m-n)\alpha}$, where we have deg $\tau_n = n(2m-n)$. In particular, $\tau_{\pm m}$ is a non-zero constant.

For a complex parameter γ, we set $\sigma \stackrel{\mathrm{def}}{=} \exp(-\gamma F)\tau$. By the definition of $\mathcal{O}$, we have $\sigma \in \mathcal{O}$. In addition, since F preserves the degree, we can write $\sigma = \sum_{n=0}^{2m} \sigma_{m-n} e^{(m-n)\alpha}$, $\deg \sigma_n = n(2m-n)$, in a similar way to τ. Now we see the coefficient σ_{-m} is constant in $\boldsymbol{t}$, and more explicitly we have

$$\sigma_{-m} = (-1)^m \tau_m \gamma^{2m} + \text{lower order term in } \gamma. \tag{5.33}$$

by (4.30). Now we can choose γ such that the polynomial σ_{-m} is equal to zero. Thus $\lambda_{-m} \notin \mathrm{supp}(\sigma)$. Then by Lemma 1, we have $S(\sigma)$ consists of the one point λ_m. That is to say $\sigma = ce^{m\alpha}$ for certain non-zero constant $c \in \mathbb{C}$. Hence we have $\tau = c\exp(\gamma F)e^{m\alpha}$. *Q.E.D.*

6 Virasoro singular vectors

Let define the operators L_n $(n \in \mathbb{Z})$ on V by

$$L_n = \frac{1}{4} \sum_{m \in \mathbb{Z}} : H_{m+n} H_n :, \tag{6.34}$$

where we define the *normal ordering* by

$$: H_m H_n := \begin{cases} H_m H_n \text{ if } m \leq n \\ H_n H_m \text{ if } m > n. \end{cases} \tag{6.35}$$

Then we have

$$[L_m, X_n] = -n X_{m+n}, \tag{6.36}$$

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0} \text{ id}. \tag{6.37}$$

Equation (6.37) means that the operators L_n gives a representation of the Virasoro algebra $\mathcal{L}$ with central charge $c = 1$. For $N \in \mathbb{Z}$, we set

$$\mathcal{S}(N) \stackrel{\text{def}}{=} \{v \in V \mid L_k v = 0 (k \geq 1),\ L_0 v = N v\}. \tag{6.38}$$

We also set $\mathcal{S}_n(N) \stackrel{\text{def}}{=} \mathcal{S}(N) \cap V_n$ $(n \in \mathbb{Z})$. Then we have $\mathcal{S}(N) = \bigoplus_{n \in \mathbb{Z}} \mathcal{S}_n(N)$. A vector v in $\mathcal{S}_n(N)$ is called a *singular vector* in V_n of *grade* N. For any $n \in \mathbb{Z}$, $e^{n\alpha} \in V_m$ is a singular vector of grade n^2. This follows from $H_k e^{n\alpha} = 0$ for $k > 0$ and $H e^{n\alpha} = 2n e^{n\alpha}$.

Now we remark that equation (6.36) implies, in particular, the following notable fact:

$$[\mathring{\mathfrak{g}}, \mathcal{L}] = 0 \quad \text{on } V. \tag{6.39}$$

Therefore, starting from a singular vector $e^{m\alpha}$, $\mathring{\mathfrak{g}}$ creates other singular vectors . By virtue of Theorem 1, they are nothing but rectangular Schur functions.

Theorem 3 *([1],[2]) If there exists a non-negative integer m such that $N = m^2$ then*

$$\mathcal{S}(N) = \bigoplus_{-m \leq n \leq m} \mathbb{C} S_{\square(m-n, m+n)}(\boldsymbol{t}) e^{n\alpha}, \tag{6.40}$$

and then $\mathcal{S}(N)$ is an irreducible $\mathring{\mathfrak{g}}$-module, otherwise $\mathcal{S}(N) = \{0\}$.

Proof. Let m be a non-negative integer. Put $N = m^2$. By the discussion above, we have

$$S_{\square(m-n, m+n)}(\boldsymbol{t}) e^{n\alpha} \in \mathcal{S}_n(N), \tag{6.41}$$

for $n \in \mathbb{Z}$ such that $-m \leq n \leq m$. The irreducibility of the right hand side of (6.40) as $\mathring{\mathfrak{g}}$-module can be seen from its character. Let $W_{m,n} \subset V$ be the $\mathcal{L}$-submodule generated by $S_{\square(m-n, m+n)}(\boldsymbol{t}) e^{n\alpha}$. Then $W_{m,n}$ is an irreducible highest weight $\mathcal{L}$-module. In fact, V

is known to be completely reducible because it is *unitarizable.* Now we can make use of the following character formula due to Kac ([13]):

$$\mathrm{tr}_{W_{m,n}} q^{L_0} = \frac{q^{m^2}(1-q^{2m+1})}{\prod_{k=1}^{\infty}(1-q^k)}. \tag{6.42}$$

Here we set $W = \bigoplus_{m=0}^{\infty} \bigoplus_{-m\le n\le m} W_{m,n}$. To complete the proof, it is suffices to note $W = V$, which is due to Frenkel [14]. Namely, we have the following character identity:

$$\begin{aligned} \mathrm{tr}_W z^H q^{L_0} &= \sum_{m=0}^{\infty} \sum_{-m\le n\le m} \mathrm{tr}_{W_{m,n}} z^H q^{L_0} \\ &= \sum_{m=0}^{\infty} \sum_{-m\le n\le m} \frac{z^{2n} q^{m^2}(1-q^{2m+1})}{\prod_{k=1}^{\infty}(1-q^k)} \\ &= \frac{\sum_{n\in\mathbb{Z}} z^{2n} q^{n^2}}{\prod_{k=1}^{\infty}(1-q^k)} = \mathrm{tr}_V z^H q^{L_0}. \end{aligned} \tag{6.43}$$

Q.E.D.

The irreducible decomposition of the Virasoro module V_n and the formulas of the singular vectors in terms of rectangular Schur functions are wellknown. These results were obtained more than 15 years ago by Segal [1] and, independently, by Wakimoto-Yamada [2]. In the present paper, we make use of the action of the affine Lie algebra on the space $V = \oplus_{n\in\mathbb{Z}} V_n$. This viewpoint enables us to explain the formula of singular vectors quite naturally.

7 Concluding remarks

Taking each of the maximal weight vectors $e^{m\alpha}$ in $L(\Lambda_0)$ as a *seed* solution, we have constructed all of the τ-function of the NLS-Toda hierarchy, which are eigenvectors of L_0. It turns out that each of them constitutes a finite Toda chain of the Schur functions associated with rectangular Young diagrams. Moreover, we have got a clear understanding that each component $\tau_m e^{m\alpha}$ of a τ-function is a singular vector of the Virasoro algebra. It will be an interesting problem to describe the τ-functions with a support on the arbitrary polyhedra other than the segment $[\lambda_{-m}, \lambda_m]$.

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